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Alternative actions for quantum gravity and the intrinsic rigidity of the spacetime

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Abstract

Using the Steiner-Weyl expansion formula for parallel manifolds and the so called gonihedric principle we find a large class of discrete integral invariants which are defined on simplicial manifolds of various dimensions. These integral invariants include the discrete version of the Hilbert-Einstein action found by Regge and alternative actions which are linear with respect to the size of the manifold. In addition the concept of generalized deficit angles appear in a natural way and is related to higher order curvature terms. These angles may be used to introduce various aspects of rigidity in simplicial quantum gravity.

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1 Introduction

String theory and theories beyond string theory have increased the interest for quantum gravity and physics at the Planck scale. One problem in these theories is the lack of a non-perturbative definition. Although string duality seems to open tantalizing possibilities for extracting non-perturbative physics from different perturbative sectors of the theory, it is still not clear that we will be able to address the full physical content at the Planck scale without a non-perturbative definition of string theory or its generalization. In particular, it is not clear how much we will be able to say about the aspects related to the quantum gravity sector of the theory.

It is therefore possible that quantum gravity, or at least important aspects of quantum gravity, may be described entirely within the framework of a non-perturbative field theory. A first step in this direction is a non-perturbative definition of the field theory we will denote quantum gravity. This is a non-trivial task because the continuum theory has to be invariant under reparametrizations. Lattice gauge theory is an example of a successful non-perturbative regularization of a quantum field theory with a continuous internal symmetry. The regularization breaks Euclidean invariance (which is restored in the scaling limit), but maintains from a certain point of view the concept of local internal invariance. In the case of gravity the situation is more difficult since we deal with local symmetries of space–time itself, and any lattice regularization will break this symmetry. In classical gravity a very natural discretization was suggested by Regge [1] by the restriction to piecewise linear manifolds, and he showed how the Einstein–Hilbert action had a natural geometric representation on the class of piecewise linear manifolds and could be expressed entirely in terms of intrinsic invariants of the piecewise linear manifolds. In this way one achieved a coordinate free description of this class of manifolds and their actions, where the dynamical variables were the link length. The use of Regge calculus as a prescription for quantum gravity is less straightforward, since there is not a one-to-one correspondence between the piecewise linear metric and the length assigned to the links (for a discussion, see [2] and also [3] and references therein). However, it is very encouraging that a variant of Regge calculus, known as dynamical triangulation [4, 5, 6], works perfect in two-dimensional quantum gravity. In this formalism one fixes the link–length of the piecewise linear manifold, and the assignment of a metric depends only on the connectivity of the triangulation. The summation over triangulations with different connectivity takes the role of integration over inequivalent metrics and the action assigned to the manifold is calculated by Regge's prescription for a piecewise linear manifold. In this formalism the scaling limit agrees with the known continuum results of two-dimensional quantum gravity [7, 8], i.e. one gets a reparametrization invariant theory in the scaling limit. While it is easy to generalize the definition of the two-dimensional discrete model to both three [9] and four dimensions [10] (see also [11] for an earlier slightly different formulation), the models can presently only be analyzed by numerical methods [12] and we have no continuum theory of Euclidean gravity in dimensions larger than two with which we can compare. Whether one uses the formalism of dynamical triangulations or the original formalism of Regge with fixed connectivity and variable link length,

it might well be that the simplest versions of discretized Einstein–Hilbert action which have been used so far at the discretized level are insufficient in producing an interesting continuum limit in dimensions larger than two. On the other hand it is very appealing to use some class of piecewise linear manifolds as the natural choice of discretization in quantum gravity since there is a one-to-one correspondence between piecewise linear structures and smooth structures for manifolds of dimensions less than seven. This motivates the search for "natural" integral invariants which are defined on piecewise linear manifolds.

In this article we would like to advocate a geometrical way to construct integral invariants on a simplicial manifold. It contains Regge's result as a special case and allows us to construct a large class of new integral invariants which might be helpful in the attempts to define a regularized path integral in quantum gravity which possesses an interesting continuum limit.

The method is based on Steiner's idea of a *parallel manifold* [15, 16, 17]. Let M_{n-1} be a smooth $(n$–$1)$-dimensional manifold embedded in n-dimensional flat space E^n, and let M^{ρ}_{n-1} denote the so called parallel manifold defined by displacing each point P the distance ρ along the outward normal at P. The hyper-volume of the parallel manifold M^{ρ}_{n-1} can be expanded into a polynomial of the distance ρ from the original manifold . The coefficient to ρ^k in this expansion represents an integral invariant $\mu_k(M_{n-1})$, $k = 0, 1, ..., n$–1 constructed on the $(n$–$1)$-dimensional manifold M_{n-1} by means of the the intrinsic and extrinsic curvature [16]. It turns out that this expansion makes perfect sense not only for smooth manifolds but also for a piecewise linear manifold and the classical invariants $\mu_k(M_{n-1})$ become natural invariants also for simplicial manifolds, in the same way as the integral of the scalar curvature in the classical work of Regge has a natural geometric definition directly on the piecewise linear manifolds. In fact the Einstein–Hilbert action is equal $\mu_2(M_{n-1})$ and therefore among the classical invariants.

An important observation emerging from the Steiner expansion for piecewise linear manifolds is that in all cases the integral invariants $\mu_k(M_{n-1})$ are the product of the *volume* of the faces of M_{n-1} and the *volume* of the corresponding *normal images* or *spherical image* of these faces (to be defined precisely in the next section):

$$\text{integral invariant} = \sum_{\{faces\}} (\text{volume of face}) \cdot (\text{volume of spherical image}). \qquad (1)$$

In this equation the notation "face" means vertex, edge, triangle, tetrahedron and higher dimensional sub-simplexes of M_{n-1}.

The generalization of the factorization property (1) of the classical integral invariants in the form of gonihedric principle allows to construct new integral invariants on a manifold M_{n-1}. In accordance with this principle one should always multiply the *volume* of the face by one of the *geometric measures*, i.e. length, area, volumes etc, associated with the corresponding spherical image:

$$\sum_{\{faces\}} (\text{volume of face}) \cdot (\text{geometric measure on spherical image}). \qquad (2)$$

This form of extension of classical integral invariants (1) maintain the locality of the

integral invariants and allow to construct a large class of discrete integral invariants defined on triangulated manifolds.

Performing a duality transformation it is possible to obtain a dual form of the new invariants

$$\sum_{\{faces\}} (\text{geometric measure on face}) \cdot (\text{volume of spherical image}). \tag{3}$$

It is also useful to consider more general integral invariants, obtained from (3) by replacing the volume density of the spherical image with a function θ(volume on image) of the volume density of the spherical image, where the choice of function θ will be dictated by the specific physical problem under consideration [18].

Since the spherical images are defined by the embedding of the manifold into Euclidean space it is not an easy task to understand why some of the integral invariants are nevertheless intrinsic and independent of the embedding. In fact, Gauss himself was very surprised when he discovered that the "Gaussian curvature" which, in analogy with the curvature of a path, was defined by means of the spherical image, was indeed a *bending invariant*. This is a "Theorema egregium", a "most excellent theorem", wrote Gauss. In modern language we understand that the underlying reason is that for smooth manifolds some of the integral invariants are constructed from the Riemann tensor and thus are independent of their embedding in Euclidean space. For the discrete version of the classical invariants and for the new ones considered here this can be seen applying the Gauss-Bonnet theorem to the spherical images [21, 22, 23, 26] and expressing the measures on the image in terms of the internal angles .

In particular, in four dimensions, this approach allows to obtain natural discrete representation of the gravitational action which contains terms quadratic in curvature tensor. The Hilbert-Einstein action together with the higher derivative terms take the form

$$\begin{aligned} Action \;=\; & \frac{1}{G}\sum_{\langle ijk\rangle} \sigma_{ijk}\cdot\omega^{(2)}_{ijk} + g_1 \sum_{\langle ijk\rangle} \omega^{(2)}_{ijk} + g_2 \sum_{\langle ijk\rangle} \left(\omega^{(2)}_{ijk}\right)^2 + \\ & + k_1 \sum_{\langle i\rangle} (1-\frac{1}{2\pi^2}\Omega^{(4)}_i) + k_2 \sum_{\langle i\rangle} \left(1-\frac{1}{2\pi^2}\Omega^{(4)}_i\right)^2 + \cdots, \end{aligned} \tag{4}$$

where σ_{ijk} is the area of the triangle $< ijk >$, $\omega^{(2)}_{ijk}$ is the deficit angle associated with the same triangle and the first term is just the Regge action of the piecewise linear manifold. The terms associated with the coupling constants g_i are functions of the deficit angles $\omega^{(2)}_{ijk}$ and in this way they corresponds to higher derivative terms. The quantity $\Omega^{(4)}_i$ is the total solid angle associated with the vertex i, i.e. $\Omega^{(4)}_i = \sum \Omega^{(4)}_{ijklm}$, where $\Omega^{(4)}_{ijklm}$ is the solid angle at the vertex of the four dimensional tetrahedron $\langle ijklm\rangle$ and the summation is over adjacent tetrahedra. The terms associated with the k_i's represent functions of the "solid deficit angles" at each vertex i. As we shall see these terms are likewise related to higher derivative terms. In a discretized approach to quantum gravity, where the functional integration is first defined by

integration over a suitable set of piecewise linear manifolds, and the continuum limit is taken afterwards, it is natural to include these terms. They will introduce an intrinsic rigidity into simplicial quantum gravity, which might be needed if one should be able to define an interesting continuum limit simplicial quantum gravity.

In the next section we shall review the Steiner-Weyl expansion method and derive the discrete version of the classical invariants. In subsequent sections we shall construct new extensions of these functionals in two, three, four and high dimensions and apply this approach to get discrete version of the high derivative terms in quantum gravity.

2 Integral invariants on smooth manifold

Let M_{n-1} be a compact orientable hyper-surface embedded in an Euclidean space E^n by a map $P \mapsto X_\mu(P)$. At a point P of M_{n-1} there are n–1 principal curvatures R_i, $i = 1,..,n$–1. If dv_{n-1} denotes the element of hyper-volume of M_{n-1}, the integral invariants of M_{n-1} can be defined as a integrals over symmetric functions of the principal radii of curvature R_i [16]

$$\mu_k(M_{n-1}) \equiv \int \{\frac{1}{R_{i_1}}...\frac{1}{R_{i_k}}\} dv_{n-1}, \tag{5}$$

where $k = 0, 1, .., n$–1 and $\{...\}$ denotes the symmetrization . In particular, μ_0 is the hyper-volume, μ_2 is the Hilbert-Einstein action and μ_{n-1} is the *degree* of the so called normal map

$$P \in M_{n-1} \mapsto n(P) \in S^{n-1},$$

which maps a point P of the manifold M_{n-1} into the unit vector $n(P)$ normal to M_{n-1}. Let Ω be a subset of M_{n-1}. Then $n(\Omega)$ is called the *spherical image* of Ω. If $d\omega_{n-1}$ denotes the hyper-volume element on S^{n-1}, then

$$dv_{n-1} = R_1...R_{n-1} d\omega_{n-1} \tag{6}$$

where the product $1/(R_1...R_{n-1})$ is the Gauss-Kronecker curvature. This is the generalization of the famous two-dimensional formula and one can use this formula to express the integral invariants (5) of M_{n-1} as integrals over the spherical image of M_{n-1}:

$$\mu_k(M_{n-1}) = \int \{R_{i_1}...R_{i_{n-k-1}}\} d\omega_{n-1}, \tag{7}$$

where $k = 0, 1, ..., n$–1. It is now easy to see that μ_{n-1} is the degree of map $n : M_{n-1} \mapsto S^{n-1}$ defined by the field of normals.

The integral invariants (5) and (7) appear in a natural way in the so called *Steiner-Weyl expansion formula* for the hyper-volume of the parallel manifold M^ρ_{n-1} [15, 16]. If M_{n-1} is embedded in E^n we define a parallel manifold M^ρ_{n-1} as a set of all points at a distance ρ from M_{n-1}, i.e. by the map $P \mapsto X_\mu(P) + \rho \cdot n_\mu(P)$, where $n(P)$ again denotes the outward normal at P. For ρ sufficiently small the hyper-volume $\mu_0(M^\rho_{n-1})$ of the parallel manifold is equal to

$$\mu_0(M^\rho_{n-1}) = \int (R_1 + \rho)...(R_{n-1} + \rho) d\omega_{n-1}, \tag{8}$$

simply because the n–1 principal curvatures for the parallel manifold M^{ρ}_{n-1} are equal to $R_i + \rho, \quad i = 1, 2, .., n$–1. Expanding the product of the integrand we get

$$\mu_0(M^{\rho}_{n-1}) = \mu_0(M_{n-1}) + \rho \cdot \mu_1(M_{n-1}) + \rho^2 \cdot \mu_2(M_{n-1}) + ... + \rho^{n-1} \cdot \mu_{n-1}(M_{n-1}), \quad (9)$$

thus generating the whole sequence of integral invariants which we discussed above.

It can be shown that the coefficients $\mu_{2k}(M_{n-1})$ to the even powers of ρ in the expansion (9) are independent of the embedding [16], while the coefficients $\mu_{2k+1}(M_{n-1})$ to the odd powers of ρ refer explicitly to the extrinsic geometry.

3 Integral invariants on piecewise linear manifolds

The idea of a parallel manifold allows us to define the discrete versions of the integral invariants (5) and (7) for piecewise linear manifolds and in this way generalize the work of Regge.

First we consider two-dimensional surfaces [17]. Let D_3 denote a three-dimensional piecewise linear manifold in E^3 with a connected boundary. The boundary of D_3 will be a piecewise linear surface M_2 embedded in E^3. We define angles between edges (one simplexes) $\langle ij\rangle$ and $\langle ik\rangle$ as $\beta_{ij;ik}$ and between triangles (two simplexes) $\langle ijl\rangle$ and $\langle ijk\rangle$ as $\alpha_{ijl;ijk}$. These angles completely define the internal and external geometry of a surface. It is easier to use the shorthand notation

$$\beta_{ij;ik} \equiv \beta_i \qquad \text{for internal angles} \qquad (10)$$
$$\alpha_{ijl;ijk} \equiv \alpha_{ij} \qquad \text{for external angles} \qquad (11)$$

and if needed one can recover the whole notation, see Fig. 1.

Note that the parallel surface M^{ρ}_2 is *one time differentiable* even if M_2 is only piecewise linear. The area of M^{ρ}_2 contains separate parts which we shall compute. The first part, $S(M_2)$, is equal to the area of the original surface M_2, i.e. it originate from parallel displacement of the triangles $\langle ikj\rangle$ constituting M_2 the distances $\rho \cdot n(\langle ijk\rangle)$, where $n(\langle ijk\rangle)$ denotes the outward normal of triangle $n(\langle ijk\rangle)$ in the orientable triangulation of M_2:

$$S(M_2) = \sum_{\langle ijk\rangle} \sigma_{ijk} \cdot 1, \qquad (12)$$

where σ_{ijk} is the area of the triangle $\langle ijk\rangle$ and the summation is extended over all triangles of the surface M_2. The second part of the area, $\rho A(M_2)$, of the parallel surface is equal the sum of the areas of the cylinders surrounding the edges [17], i.e. proportional to the displacement ρ, and we can write

$$\rho \cdot A(M_2) = \rho \cdot \sum_{\langle ij\rangle} \lambda_{ij} \cdot (\pi - \alpha_{ij}), \qquad (13)$$

where λ_{ij} is the length of the edge $\langle ij\rangle$ and α_{ij} is the dihedral angle at the edge $\langle ij\rangle$ and the summation is over all edges. The third part of the area of M^{ρ}_2, $\rho^2\chi(M_2)$, is

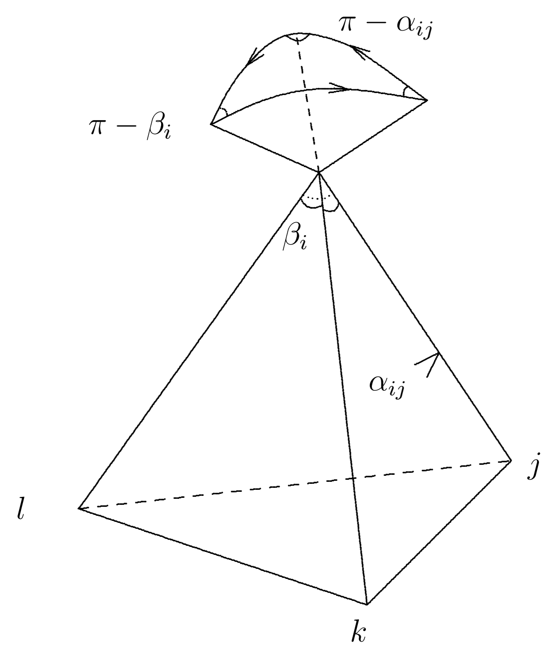

Figure 1: The spherical image of the vertex $\langle i \rangle$. The internal angle between edges are denoted as β_i and the external angles between triangles as α_{ij}.

equal to the sum of the areas of the spherical polygons which surround the vertices of M_2, i.e. proportional to ρ^2, and we can write

$$\rho^2 \cdot \chi(M_2) = \rho^2 \cdot \sum_{\langle i \rangle} 1 \cdot (2\pi - \sum \beta_i), \tag{14}$$

where $\chi_i = 2\pi - \sum \beta_i$ is the area of the spherical polygon on a unit sphere corresponding to a vertex $\langle i \rangle$, usually called a deficit angle, and the summation is over all vertices of M_2.

Comparing these quantities with the Steiner expansion (9) for smooth two-dimensional surfaces

$$S(M_2^\rho) = \int_{M_2} R_1 R_2 d\omega + \rho \cdot \int_{M_2} (R_1 + R_2) d\omega + \rho^2 \cdot \int_{M_2} d\omega, \tag{15}$$

we can get the natural discrete representation of the integral invariants for the two-dimensional surface [25]

$$\begin{aligned} S(M_2) &= \sum_{\langle ijk \rangle} \sigma_{ijk} \cdot 1, & (16)\\ A(M_2) &= \sum_{\langle ij \rangle} \lambda_{ij} \cdot (\pi - \alpha_{ij}), & (17)\\ \chi(M_2) &= \sum_{\langle i \rangle} 1 \cdot (2\pi - \sum \beta_i), & (18) \end{aligned}$$

see Fig. 1.

4 Factorization and the gonihedric principle

The important message we get from the Steiner expansion and from the formulae above is that the integral invariants are (the sum of) products of a volume of the faces of the surface M_2 and a volume of the spherical images of these faces

$$\text{integral invariant} = \sum_{\{faces\}} (\text{volume of face}) \cdot (\text{volume of spherical image}). \tag{19}$$

Indeed the spherical images of the faces of M_2 are:

$$\begin{aligned} \text{triangle} \quad \langle ijk \rangle \quad &\rightarrow \quad \text{point on } S^2, &(20)\\ \text{edge} \quad \langle ij \rangle \quad &\rightarrow \quad \text{arc on } S^2, &(21)\\ \text{vertex} \quad \langle i \rangle \quad &\rightarrow \quad \text{spherical polygon on } S^2. &(22) \end{aligned}$$

and the volume elements on these spherical images are

$$\begin{aligned} \text{triangle } \langle ijk \rangle \quad &\rightarrow \quad 1 &(23)\\ \text{edge } \langle ij \rangle \quad &\rightarrow \quad (\pi - \alpha_{ij}) &(24)\\ \text{vertex } \langle i \rangle \quad &\rightarrow \quad (2\pi - \sum \beta_i), &(25) \end{aligned}$$

i.e. equal to one for a point in S^2 which is the image of a triangle, equal to the length of the arc in S^2 which is the image of an edge, and equal to the area of the polygon on S^2 which is the image of a vertex. They are functions of the internal and external angles (10), and (11). The factorization or goni-hedric structure (19) of the integral invariants (16)-(18) is transparent now.

The generalization of the factorization property (19) of the classical integral invariants in the form of gonihedric principle can be used now to construct new integral invariants on a manifold M_2. In accordance with this principle one should always multiply the *volume* of the face by one of the *geometric measures*, associated with the corresponding spherical image:

$$\sum_{\{faces\}} (\text{volume of face}) \cdot (\text{geometric measure on spherical image}). \tag{26}$$

In the two-dimensional case considered so far the area functional $S(M_2)$ is uniquely defined since there is no nontrivial measure associated with the point on S^2 which is the spherical image of the triangles of M_2. The same is true for the linear functional $A(M_2)$, because only the length of the arc can be associated with the spherical image (24) of the edge (21).

Only the topological invariant (18) can lead to new expressions because the spherical image (22) of the vertex $\langle i \rangle$ is a polygon on S^2 (see Fig. 2), and the polygon has sufficiently structure to allow non-trivial measures. The possible purely geometric measures defined on a polygon on S^2 are (see Fig. 2):

$$\begin{aligned} \text{the area} \quad &= \quad (2\pi - \sum \beta_i), &(27)\\ \text{the curvature} \quad &= \quad \sum (\pi - \beta_i), &(28)\\ \text{the perimeter} \quad &= \quad \sum_j (\pi - \alpha_{ij}). &(29) \end{aligned}$$

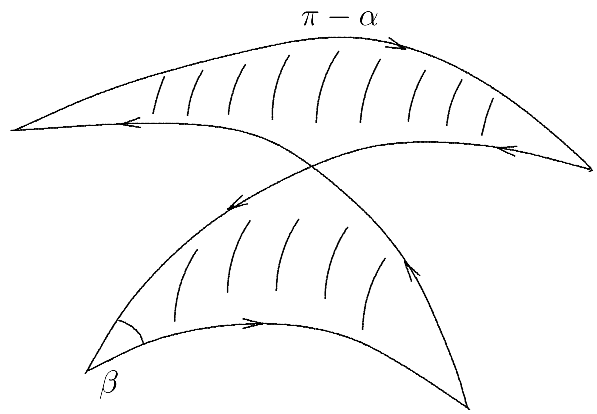

Figure 2: On every spherical polygon one can define the area (27), the perimeter (28) and the curvature (29).

As we have seen the total area of the spherical image is proportional to the Euler character (see (18), (25) and (27). The total curvature of the spherical polygon *cannot be used*, because it is possible by a continuous deformation of M_2 to ensure that there is zero curvature (no deficit angle) associated with a given vertex, in which case the polygon image (22) will disappear but it will leave a nonzero curvature (28). Such a situation is not physical interesting and we exclude it. This means that only the perimeter (29) can be used in the construction of new invariants, and in accordance with (26) we define

$$\Theta(M_2) = \sum_{\langle i \rangle} \Big(\sum (\pi - \alpha_{ij}) \Big). \tag{30}$$

Every term with dihedral angle α_{ij} appears in both vertices $\langle i \rangle$ and $\langle j \rangle$ therefore after resummation one get

$$\Theta(M_2) = 2 \sum_{\langle ij \rangle} (\pi - \alpha_{ij}). \tag{31}$$

The trivial resummation of quantities given on a vertices into a sum over adjacent edges will again appear in high dimensions but in the less trivial form of a dual transformation.

The invariant (31) is equal to the total length of the arcs of the spherical image of all edges of the surface M_2. This quantity is a natural definition of the total *extrinsic curvature* of the triangulated surface. There is a simple mnemonic rule by which one can get this integral invariant out of the formulae already available, i.e. (16)-(18). Indeed, if we shall consider a triangulated surface with fixed length of all edges $\lambda_{ij} = a$ (as in the case of dynamical triangulated surfaces), then the functional $A(M_2)$ is proportional to $\Theta(M_2)$

$$A(M_2) = a \cdot \Theta(M_2). \tag{32}$$

Let us finally consider generalizations where

$$\text{measure on spherical image} \to \theta(\text{measure on spherical image})$$

as discussed above. The obvious and simplest example is

$$\pi - \alpha_{ij} \to |\pi - \alpha_{ij}|.$$

In this way we can define a new integral invariant

$$\tilde{\Theta}(M_2) = \sum_{\langle ij \rangle} |\pi - \alpha_{ij}|, \tag{33}$$

which is a natural discretized version of

$$\int_{M_2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^2 dS. \tag{34}$$

It is also possible to consider more general functions [18]

$$\pi - \alpha \to \theta(\alpha), \qquad \theta(\pi) = 0 \tag{35}$$

where $\theta(\alpha)$ increases monotonically at the both sides from the point $\alpha = \pi$ and near that point has the following parametrization

$$\theta(\alpha) = (\pi - \alpha)^{\varsigma}. \tag{36}$$

For these more general functions (33) is replaced by

$$\tilde{\Theta}(M_2) = \sum_{\langle ij \rangle} \theta(\alpha_{ij}). \tag{37}$$

The index ς (36) is important for the convergence of the corresponding partition function and the critical properties of the system [18, 19].

In order to demonstrate the value of these invariants for the theory of random surfaces, let us compute them for a sphere of radius R which is triangulated in the following way: first we divide it in n degrees of latitude and m degrees of longitude and each square obtained this way is divided in two triangles. With this division we get for the action $A(M_2)$ defined by (17):

$$A(S^2; \varsigma) = 4\pi R \left((\frac{n}{a})^{1-\varsigma} + (\frac{m}{b})^{1-\varsigma} \right) \tag{38}$$

and for the action $\tilde{\Theta}(M_2)$ defined by (37):

$$\tilde{\Theta}(S^2; \varsigma) = anm^{1-\varsigma} + bmn^{1-\varsigma}, \tag{39}$$

where a and b are constants. We have different behavior of the action in the limit $n, m \to \infty$. If $\theta(\alpha) = 1 - \cos\alpha$, i.e. $\varsigma = 2$, we get

$$\tilde{\Theta}(S^2; \varsigma = 2) = \pi^2 \frac{m}{n} + 2\pi \frac{n}{m}. \tag{40}$$

Thus the index ς influence in an essential way the convergence of the partition function, and should be chosen less than one in order to have an interesting theory.

5 Three-dimensional manifolds

We consider now a four-dimensional domain D_4 in E^4 bounded by the three-dimensional manifold M_3. The three-volume of the parallel manifold M_3^ρ has the form (9), i.e.

$$\begin{aligned} V(M_3^\rho) &= \int_{M_3} R_1 R_2 R_3 d\omega_3 + \rho \cdot \int_{M_3} \{R_1R_2 + R_2R_3 + R_3R_1\} d\omega_3 \\ &\quad + \rho^2 \cdot \int_{M_3} \{R_1 + R_2 + R_3\} d\omega_3 + \rho^3 \cdot \int_{M_3} d\omega_3 \end{aligned} \tag{41}$$

or in terms of notation used so far

$$V(M_3^\rho) = V(M_3) + \rho \cdot S(M_3) + \rho^2 \cdot A(M_3) + \rho^3 \cdot N(M_3) \tag{42}$$

which defines the volume, the area, the linear size and the topology of the domain.

We now apply the method of parallel expansion to the piecewise linear three-dimensional manifolds to get discrete versions of the corresponding integral invariants. Following the procedure in two dimensions we introduce angles between one, two and three simplexes in order to define the internal and external geometry of the simplicial manifold (see Fig. 3):

$$\beta_{ij;ik} \equiv \beta_i \qquad \text{for internal angles} \tag{43}$$

$$\beta_{ijl;ijk} \equiv \beta_{ij} \qquad \text{for internal angles} \tag{44}$$

$$\alpha_{ijkl;ijkm} \equiv \alpha_{ijk} \qquad \text{for external angles} \tag{45}$$

and we introduce the corresponding spherical images of the simplexes of M_3 as

$$\text{tetrahedron } \langle ijkl \rangle \quad \to \quad \text{point on } S^3 \tag{46}$$

$$\text{triangle } \langle ijk \rangle \quad \to \quad \text{arc on } S^3, \tag{47}$$

$$\text{edge } \langle ij \rangle \quad \to \quad \text{polygon on } S^3, \tag{48}$$

$$\text{vertex } \langle i \rangle \quad \to \quad \text{spherical polyhedron on } S^3 \tag{49}$$

Using the angles already defined on the simplexes (43)- (45) one can find the geometric measures on the spherical images (46)-(49):

$$\text{tetrahedron } \langle ijkl \rangle \quad \to \quad 1 \tag{50}$$

$$\text{triangle } \langle ijk \rangle \quad \to \quad (\pi - \alpha_{ijk}) \tag{51}$$

$$\text{edge } \langle ij \rangle \quad \to \quad (2\pi - \sum \beta_{ij}) \tag{52}$$

$$\text{vertex } \langle i \rangle \quad \to \quad \omega_i^{(3)}(\alpha, \beta) \tag{53}$$

Using this information it is not difficult to compute the volume of the parallel manifold. It will contain four parts which we should identify with invariants (42) of the simplex M_3. If v_{ijkl} denotes the volume of the tetrahedron $\langle ijkl \rangle$ and $\omega_i^{(3)}$ denotes the volume in S^3 of the spherical polyhedron corresponding to vertex $\langle i \rangle$,

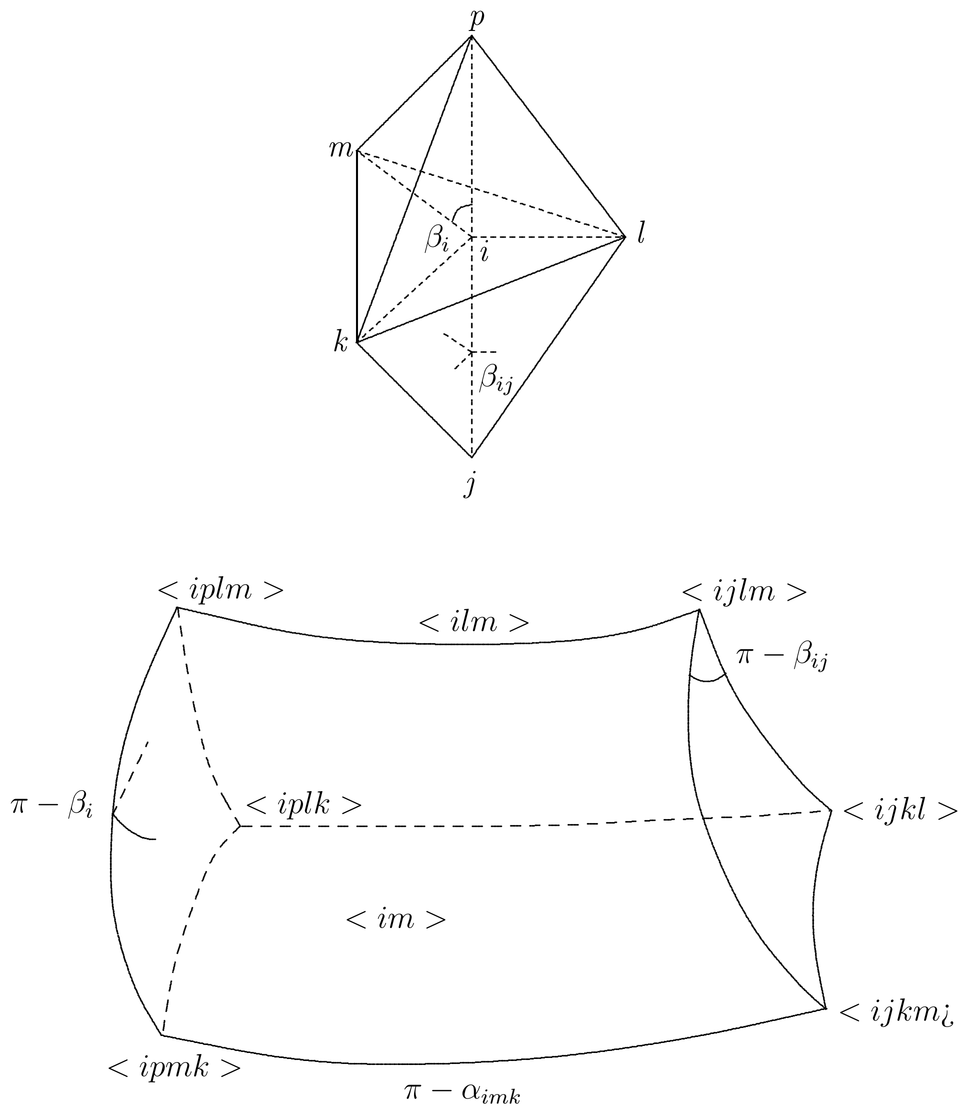

Figure 3: The vertex of the three-dimensional simplex (shown on the upper figure) and the corresponding spherical polyhedron on S^3. The volume of this image is denotes $\omega_i^{(3)}$, the area as $\sum \omega_{ij}^{(2)}$, where the summation is over all polygons of the given polyhedron, and finally the linear size as $\sum \Omega_{ijk}^{(2)} \omega_{ijk}^{(1)}$ and the summation is over all arcs of the polyhedron

we obtain:

$$\begin{aligned}
\text{the volume } V(M_3) &= \sum_{\langle ijkl\rangle} v_{ijkl}\cdot 1 &(54)\\
\text{the area } S(M_3) &= \sum_{\langle ijk\rangle} \sigma_{ijk}\cdot(\pi-\alpha_{ijk}), &(55)\\
\text{the linear function } A(M_3) &= \sum_{\langle ij\rangle} \lambda_{ij}\cdot(2\pi-\sum\beta_{ij}), &(56)\\
\text{the topological invariant } N(M_3) &= \sum_{\langle i\rangle} 1\cdot\omega_i^{(3)}(\alpha,\beta). &(57)
\end{aligned}$$

All these integral invariants have the factorized form (19).

Let us now consider the possible new integral invariants along the lines already discussed in the two-dimensional case. Again the spherical images of the two highest dimensional simplexes, the tetrahedron and the triangle are of lowest dimension, i.e. a point and an arc, respectively, and it is impossible to find any geometric extension for these objects. The first non-trivial simplex is from this point of view the edge, which has as its spherical image a polygon. For the polygon we can in addition to its area also use its perimeter as a geometric measure, and hence we find a linear invariant different from $A(M_3)$:

$$\Lambda(M_3)=\sum_{\langle ij\rangle}\lambda_{ij}\cdot\Big(\sum(\pi-\alpha_{ijk})\Big), \tag{58}$$

in accordance with our principle (26). The spherical image of a vertex $\langle i\rangle$ is a spherical polyhedron (49), and for this object we have the topological invariant (57), which corresponds to the use of the three-volume $\omega_i^{(3)}$ as the geometric measure. However, we can also use the arc–length or the area of the two–simplexes in the polyhedron as geometric measure (see Fig. 3), i.e.:

$$\sum\ \beta_i\ (\pi-\alpha_{ijk}) \quad \text{or} \quad \sum\ (2\pi-\sum\beta_{ij}). \tag{59}$$

This leads to the following new integral invariants

$$\Theta(M_3)=\sum_{\langle i\rangle}1\cdot\Big(\sum\beta_i\ (\pi-\alpha_{ijk})\Big) \tag{60}$$

$$\Xi(M_3)=\sum_{\langle i\rangle}1\cdot\Big(\sum(2\pi-\sum\beta_{ij})\Big). \tag{61}$$

5.1 The dual representation

There is an equivalent or "dual" form of the invariants (58), (60) and (61), which can be obtained after resummation. In (58) we can combine all edges belonging to a given triangle $\langle ijk\rangle$ to get a sum $\lambda_{ij}+\lambda_{jk}+\lambda_{ki}=\lambda_{ijk}$. Hence,

$$\Lambda(M_3)=\sum_{\langle ijk\rangle}\lambda_{ijk}\cdot(\pi-\alpha_{ijk}) \tag{62}$$

where α_{ijk} is the angle between two neighboring tetrahedra of M_3 having a common triangle $\langle ijk \rangle$ of the perimeter λ_{ijk}. This is the dual form of the integral invariant (58) since either one can multiply the length of the edge by the total perimeter of the polygon which is the spherical image of the edge, or one can multiply the perimeter of the triangle by the length of the arc of the spherical image of the triangle, as in (62).

The same kind of transformations, in the form of a resummation of the quantities given at the vertices and in this way transferring the sum to one over adjacent edges or triangles, work for the two invariants (60) and (61). Collecting the terms belonging to the same triangle $\langle ijk \rangle$ in (60) we can get $(\beta_i + \beta_j + \beta_k)(\pi - \alpha_{ijk}) = \pi(\pi - \alpha_{ijk})$ therefore

$$\Theta(M_3) = \sum_{\langle ijk \rangle} (\pi - \alpha_{ijk}). \tag{63}$$

Note that $\Theta(M_3)$ is the total extrinsic curvature of M_3. If we collect the terms belonging to the same edge in (61) we simply get

$$\Xi(M_3) = \sum_{\langle ij \rangle} (2\pi - \sum \beta_{ij}). \tag{64}$$

$\Xi(M_3)$ is the total deficit angle of M_3.

The gonihedric principle (26) which we used to construct extensions can now be formulated in *dual* form: either one should multiply the volume of the face by all possible geometric measures which can be constructed on its spherical image, or one should multiply all possible geometric measures on the face by the volume of its spherical image.

Finally, we remark that also in the three-dimensional case one can of course introduce more general invariants by considering functions θ of the geometric measures on the spherical images, as was done in the two-dimensional case.

6 Four-dimensional manifolds

The Steiner expansion for the hyper-volume Ω of a smooth manifold M_4 reads

$$\Omega(M_4^\rho) = \Omega(M_4) + \rho \cdot V(M_4) + \rho^2 \cdot S(M_4) + \rho^3 \cdot A(M_4) + \rho^4 \cdot \chi(M_4) \tag{65}$$

and by comparison with the same expression for a piecewise linear manifold M_4 we get the discrete versions of the above invariants

$$\Omega(M_4) = \sum_{\langle ijklm \rangle} v_{ijklm} \cdot 1, \tag{66}$$

$$V(M_4) = \sum_{\langle ijkl \rangle} v_{ijkl} \cdot (\pi - \alpha_{ijkl}), \tag{67}$$

$$S(M_4) = \sum_{\langle ijk \rangle} \sigma_{ijk} \cdot (2\pi - \sum \beta_{ijk}), \tag{68}$$

$$A(M_4) = \sum_{\langle ij \rangle} \lambda_{ij} \cdot \omega_{ij}^{(3)}(\alpha, \beta), \tag{69}$$

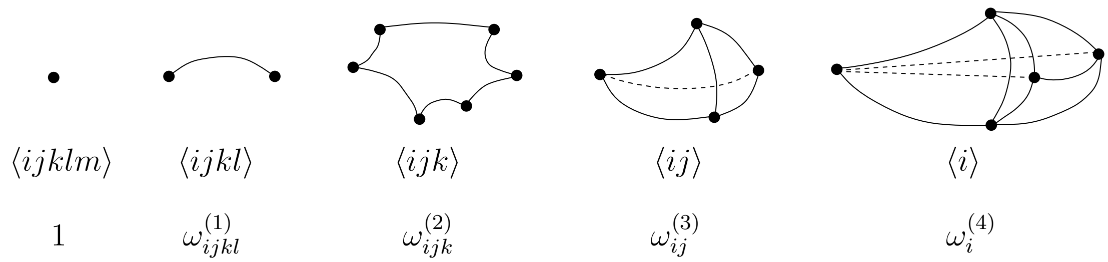

Figure 4: The sub-simplexes of the four-dimensional manifold (second line) and the corresponding spherical images on S^4 (first line). The hyper-volume of the image is defined as ω^4_i, the volume of the image as $\sum \omega^{(3)}_{ij}$, the area of the image as $\sum \Omega^{(2)}_{ijk}\omega^{(2)}_{ijk}$, and finally the length of the image as $\sum \Omega^{(3)}_{ijk}\omega^{(1)}_{ijkl}$.

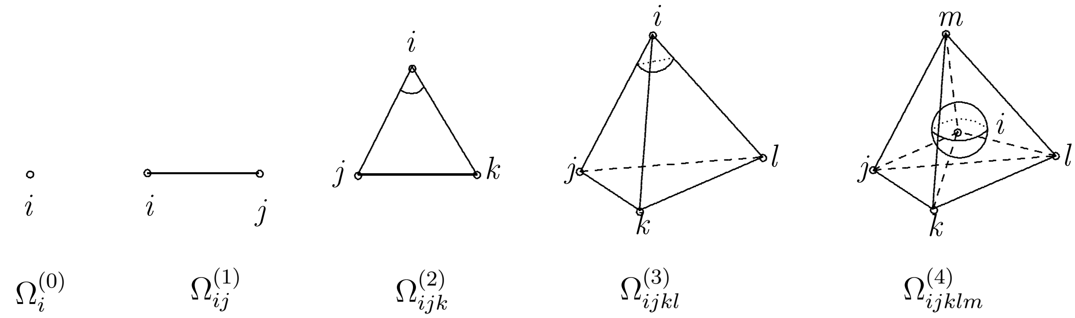

Figure 5: Solid angles of the faces of a vertex and the geometric measures.

$$\chi(M_4) \;=\; \sum_{\langle i \rangle} 1 \cdot \omega^{(4)}_i(\beta), \tag{70}$$

where as before we introduce internal angles $\beta_i, \beta_{ij}, \beta_{ijk}$ between one, two and three simplexes and α_{ijkl} for the external angle between two four-dimensional simplexes, as well as the notation v_{ijklm} for the four-volume of the four-simplex $\langle ijklm \rangle$ and the notation $\omega^{(4)}_i$ for the four-volume in S^4 of the spherical image of the vertex $\langle i \rangle$ (see Fig. 4). Note that ω^4_i is independent of the external angle α. This is in agreement with the general pattern already mentioned which states that the coefficients to even powers of ρ are intrinsic integral invariants, while the coefficients to the odd powers of ρ contain reference to the extrinsic geometry. The whole four-volume (70) obtained by summation over all vertices of the piecewise linear manifold M_4 is proportional to The Euler-Poincare character in the same way as in the two-dimensional case.

To analyze the possible new integral invariants in four and higher dimensions, we need to introduce a universal notation for *solid angles* on the faces of a simplex and on the corresponding spherical images. We shall use $\Omega^{(k)}$ for solid angle at a vertex of a k-dimensional simplex (see Fig. 5) and $\omega^{(k)}$ for the solid angles on spherical images (see Fig. 4). All these solid angles are functions of the previously introduced angles denoted bye α's and β's with various indices, see Fig. 5.

The possible geometric measures on the spherical image of the vertex $\langle i \rangle$ are:

hyper-volume, volume, area and length [26, 21, 22, 23], (see Fig. 4):

$$\omega_i^{(4)}, \qquad \sum \omega_{ij}^{(3)}, \qquad \sum \Omega_{ijk}^{(2)}\,\omega_{ijk}^{(2)}, \qquad \sum \Omega_{ijkl}^{(3)}\,\omega_{ijkl}^{(1)}. \tag{71}$$

Using these measures there is one new invariant of the "area" type (like $S(M_4)$), namely

$$\mu_S(M_4) = \sum_{\langle ijk\rangle} \sigma_{ijk} \cdot (\ \sum\ \omega_{ijkl}^{(1)}\ ). \tag{72}$$

There are two new invariants of "length" type (i.e. like $A(M_4)$):

$$\mu_A(M_4) = \sum_{\langle ij\rangle} \lambda_{ij} \cdot \left\{ \begin{array}{c} \sum\ \omega_{ijk}^{(2)} \\ \sum\ \Omega_{ijkl}^{(3)}\ \omega_{ijkl}^{(1)}, \end{array} \right\} \tag{73}$$

and finally three new invariants which are dimensionless, (like $\chi(M_4)$):

$$\mu_\chi(M_4) = \sum_{\langle i\rangle} 1 \cdot \left\{ \begin{array}{c} \sum\ \omega_{ij}^{(3)} \\ \sum\ \Omega_{ijk}^{(2)}\ \omega_{ijk}^{(2)} \\ \sum\ \Omega_{ijkl}^{(3)}\ \omega_{ijkl}^{(1)} \end{array} \right\} \tag{74}$$

In the next section we shall perform the dual transformation of the above invariants and address the question of their internal and external properties.

6.1 Dual representations

To get the dual form of the above invariants we combine terms in the parentheses belonging to the same tetrahedron in the first, third and sixth of the invariants in (72)–(74):

$$\left\{ \begin{array}{c} \sigma_{ijkl} \\ \lambda_{ijkl} \\ \Omega_{ijkl} \end{array} \right\} = \sum_{over\ \ tetrahedron\ \ \langle ijkl\rangle} \left\{ \begin{array}{c} \sigma_{ijk} \\ \lambda_{ij}\ \Omega_{ijkl}^{(3)} \\ \Omega_{ijkl}^{(3)} \end{array} \right\}. \tag{75}$$

Hence, we have the following sequence of invariants constructed from $\omega_{ijkl}^{(1)}$ associated with tetrahedron $\langle ijkl\rangle$:

$$\mu_\omega(M_4) = \sum_{\langle ijkl\rangle} \left\{ \begin{array}{c} v_{ijkl} \\ \sigma_{ijkl} \\ \lambda_{ijkl} \\ \Omega_{ijkl} \end{array} \right\} \cdot \omega_{ijkl}^{(1)}, \tag{76}$$

where $\omega_{ijkl}^{(1)} = \pi - \alpha_{ijkl}$ is the angle between two neighboring four-dimensional simplexes having a common tetrahedron $\langle ijkl\rangle$ of volume v_{ijkl}, area σ_{ijkl}, length λ_{ijkl} and total internal solid angle Ω_{ijkl}.

Combining the second and the fifth term in the parentheses in (72) and (74) belonging to the same triangle

$$\left\{ \begin{array}{c} \lambda_{ijk} \\ \pi \end{array} \right\} = \sum_{over\ \ triangle\ \ \langle ijk\rangle} \left\{ \begin{array}{c} \lambda_{ij} \\ \Omega_{ijk}^{(2)} \end{array} \right\} \tag{77}$$

we get integral invariants constructed from $\omega^{(2)}_{ijk}$'s associated to triangles ijk:

$$\mu_\omega(M_4) = \sum_{\langle ijk \rangle} \left\{ \begin{array}{c} \sigma_{ijk} \\ \lambda_{ijk} \\ \pi \end{array} \right\} \cdot \omega^{(2)}_{ijk}, \tag{78}$$

where $\omega^{(2)}_{ijk} = 2\pi - \sum \beta_{ijk}$ is the deficit angle on the triangle $\langle ijk \rangle$ which have the area σ_{ijk}, the perimeter λ_{ijk} and as usually the sum of internal angles equal to π. The first invariant in this family coincides with discrete version of the Hilbert-Einstein action [1], the second coincides with the linear action suggested in [14] and the last one is equal to the total deficit angle of the whole manifold [10].

The fourth invariant is simply equal to the volume of the spherical image of the edge and associated to the linear term (69) we have two integral invariants constructed from the three-volume of the spherical image $\omega^{(3)}_{ij}$ of the edge $\langle ij \rangle$:

$$\mu_\omega(M_4) = \sum_{\langle ij \rangle} \left\{ \begin{array}{c} \lambda_{ij} \\ 1 \end{array} \right\} \cdot \omega^{(3)}_{ij}. \tag{79}$$

To understand why some of the integral invariants introduced above are intrinsic and are independent of the embedding one should use the Gauss-Bonnet theorem. The Gauss-Bonnet theorem provides us with a general relations between the volume of the spherical image of the vertex and the solid angles associated to the vertex. Indeed, the Gauss-Bonnet theorem for the two-dimensional triangulated surface can be formulated in the form [22, 23, 24, 26]

$$\frac{1}{4\pi}\omega^{(2)}_i + \frac{1}{4\pi}\Omega^{(2)}_{ijk} = \frac{1}{2}. \tag{80}$$

Hence, summing over all vertices in the triangulation we get

$$\chi(M_2) \equiv \frac{1}{2\pi}\sum_{\langle i \rangle} \omega^{(2)}_i = \sum_{\langle i \rangle}\{1 - \frac{1}{2\pi}\sum \Omega^{(2)}_{ijk}\} = N_0 - N_2/2, \tag{81}$$

where N_0 is the number of vertices and N_2 the number of triangles on the surface M_2.

This discussion can be generalized to four dimensions: in every vertex of the simplicial manifold M_4 we have [20, 22, 23, 24, 26, 27]

$$\frac{3}{8\pi^2}\omega^{(4)}_i + \frac{1}{8\pi^2}\Omega^{(2)}_{ijk}\ \omega^{(2)}_{ijk} + \frac{1}{4\pi^2}\Omega^{(4)}_{ijklm} = \frac{1}{2}, \tag{82}$$

which shows that the hyper-volume $\omega^{(4)}_i$ on S^4 can be expressed though the intrinsic quantities. The Euler-Poincare character is equal to

$$\chi(M_4) \equiv \frac{3}{4\pi^2}\sum_{\langle i \rangle} \omega^{(4)}_i = \sum_{\langle i \rangle}\{1 - \frac{1}{2\pi^2}\sum \Omega^{(4)}_{ijklm}\} - \frac{1}{2}\cdot\frac{1}{2\pi}\sum_{\langle ijk \rangle}\omega^{(2)}_{ijk} = N_0 - N_2/2\ + N_4 \tag{83}$$

and we obtain the relation between the total deficit angle

$$\omega^{(2)}_{tot} \equiv \frac{1}{2\pi} \sum_{\langle ijk \rangle} \omega^{(2)}_{ijk} \tag{84}$$

and the total solid deficit angle

$$\Omega^{(4)}_{tot} \equiv \sum_{\langle i \rangle} (1 - \frac{1}{2\pi^2} \Omega^{(4)}_{i}) = \sum_{\langle i \rangle} \{1 - \frac{1}{2\pi^2} \sum \Omega^{(4)}_{ijklm}\} \tag{85}$$

in the form [28]

$$\chi(M_4) = \Omega^{(4)}_{tot} - \frac{1}{2}\omega^{(2)}_{tot}. \tag{86}$$

The Euler-Poincare character is

$$\chi(M_4) = \frac{1}{128\pi^2} \int_{M_4} dv_4 \Big(R^2_{\mu\nu\lambda\rho} - 4R^2_{\mu\nu} + R^2 \Big), \tag{87}$$

and comparing with the discretized version (86) of $\chi(M_4)$ it is not unnatural to associate higher powers of the integral of curvature tensors with linear combinations of $\Omega^{(4)}_{tot}$ and $\omega^{(2)}_{tot}$, although we have no explicit identification of these two terms entirely in terms of integrals of R^2, $R^2_{\mu\nu}$ etc.

From these considerations it seems that the general discretized action in simplicial four-dimensional gravity should be of the form mentioned in the introduction:

$$\begin{aligned} Action \quad = \quad & \frac{1}{G} \sum_{\langle ijk \rangle} \sigma_{ijk} \cdot \omega^{(2)}_{ijk} + g_1 \sum_{\langle ijk \rangle} \omega^{(2)}_{ijk} + + g_2 \sum_{\langle ijk \rangle} \Big(\omega^{(2)}_{ijk}\Big)^2 + \\ & k_1 \sum_{\langle i \rangle} \Big(1 - \frac{1}{2\pi^2} \Omega^{(4)}_{i}\Big) + k_2 \sum_{\langle i \rangle} \Big(1 - \frac{1}{2\pi^2} \Omega^{(4)}_{i}\Big)^2 \end{aligned} \tag{88}$$

where the terms involving the square of the deficit angles introduce an intrinsic rigidity into the simplicial manifolds. This choice of discretized action with higher derivative terms is related, but not identical to the terms suggested in [29].

7 High dimensions

It is not difficult to extend these results to high dimensional manifolds, using the same ideas and constructions. The extension of the formulas (66)-(70) and (72)-(74) or their dual form (76)- (79) is straightforward and have the same structure as already encounted in four and lower dimensions.

One new aspect in higher dimensions is the appearance of additional intrinsic integral invariants which are of interest in gravity and membrane theory. The first one coincides with Hilbert-Einstein action

$$A(M_5) = \sum_{<ijkl>} v_{ijkl} \cdot \omega^{(2)}_{ijkl} \tag{89}$$

where v_{ijkl} is the three-volume of the tetrahedron $< ijkl >$ and $\omega^{(2)}_{ijkl}$ is the deficit angle. The second invariant is proportional to the linear size of the manifold

$$A(M_5) = \sum_{<ij>} \lambda_{ij} \cdot \omega^{(4)}_{ij} \tag{90}$$

where λ_{ij} is the length of the edge and $\omega^{(4)}_{ij}$ is the hyper-volume of the corresponding spherical image on S^5. Both measures $\omega^{(2)}_{ijkl}$ and $\omega^{(4)}_{ij}$ are expressible in terms of intrinsic angles, as we have seen in the previous section, and both measures produce an sequence of integral invariants in various dimensions. The role of the deficit angle $\omega^{(2)}_{ijkl}$ is well understood because this measure leads to the Euler character of the two dimensional surfaces (18) and makes it possible to formulate discrete versions of classical gravity

$$Action^{(d)} = \sum v^{(d-2)}\omega^{(2)} \tag{91}$$

where $v^{(d-2)}$ denotes the volume of the $d-2$ dimensional sub-simplex and $\omega^{(2)}$ the deficit angle. The role of $\omega^{(4)}$ is very similar for four and higher dimensional manifolds where it leads to the Euler-Poincare character (70) and allows us to generate a new sequence of invariants:

$$Action^{(d)} = \sum v^{(d-4)}\omega^{(4)} \tag{92}$$

which may play a role in the quantum theory of extended objects. It might be difficult to investigate the effect of these actions by analytical means, but they seem well suited for use in numerical simulations.

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